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LETTER TO THE EDITOR

Cylindrical Korteweg de Vries equation and Painlevé property

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Abstract. We demonstrate that the Lax representation for the cylindrical Korteweg de Vries equation can be obtained with the help of the Painlevé property of this equation.

Recently, Weiss *et al* (1983) have introduced what is called the Painlevé property for partial differential equations. They applied their method to the soliton equations (the KdV, sine Gordon, KP equations etc) and found, in a remarkably straightforward manner, the well known Bäcklund transformations.

In the present note we apply the technique to the so-called cylindrical KdV equation (cKdV). The equation under consideration is given by

$$u_t + 6uu_x + u_{xxx} + u/(2t) = 0. \qquad (1)$$

Introducing the potential w defined by $u = -w_x$ we find that (1) can be written as

$$w_{xt} - 6w_x w_{xx} + w_{xxxx} + w_x/(2t) = 0. \qquad (2)$$

An auto Bäcklund transformation of (2) has been given by Nakamura (1980), namely

$$\begin{aligned} (w'+w)_x &= \tfrac{1}{2}(w'-w)^2 - (x+x_1)/6t \\ (w'+w)_t &= -(x+x_1)w'_x/3t + 2w_x^2 \\ &\quad + 2(w'-w)w_{xx} + (w'-w)^2 w_x + (w'-w)/6t - (w'+w)/2t. \end{aligned} \qquad (3)$$

Nakamura (1980) obtained this result with the help of the Hirota bilinear transform method. The inverse scattering transform for the cKdV equation is discussed by Calogero and Degasperis (1978a); also the conservation laws have been given by Calogero and Degasperis (1978b). The isospectral eigenvalue problem associated with the cKdV equation $u_t + u_{xxx} + uu_x + u/(2t) = 0$ is given by

$$(tD^2 - x/12 + tu/6)V = \mu V.$$

Infinitely many commuting symmetries and constants of motion for explicitly time-dependent evolution equations with application to the cKdV equation have been studied by Oevel and Fokas (1983). We show that the results given above can be obtained with the help of the method described by Weiss *et al* (1983).

In the following we study (1) in a somewhat different form in order to include the KdV equation, namely

$$u_t + uu_x + \tfrac{1}{2}u_{xxx} + au/t = 0 \qquad (4)$$

(Tajiri and Kawamoto 1982), where a is a real parameter. If $a=0$ then we have the KdV equation. Thus our results contain those of Weiss *et al* (1983). If $a=\frac{1}{2}$, then we have the cKdV equation and if $a=1$, then (4) is called the spherical KdV equation. This equation is not integrable. As far as possible the parameter a will be arbitrary in our calculations. At the end of our calculation we find that we are forced to put $a=0$ or $a=\frac{1}{2}$ in order to find that equation (4) is integrable.

In the technique described by Weiss *et al* (1983) we consider the quantities u, x and t in the complex plane. For the sake of simplicity we do not change our notation. For the field u we make the series ansatz

$$u(x,t)=\phi^{\alpha}(x,t)\sum_{j=0}^{\infty}u_j(x,t)\phi^j(x,t). \tag{5}$$

If α is an integer and if it is possible to cut off this series expansion at a certain integer, say n ($n<\infty$), and moreover the equations for the fields $\phi, u_0, u_1, \ldots, u_n$ are compatible, then we obtain Bäcklund transformations.

Let us now perform the calculation step by step. First of all we determine the dominant behaviour, i.e. we determine the exponent α. Inserting the ansatz

$$u(x,t)\sim\phi^{\alpha}(x,t)u_0(x,t) \tag{6}$$

into (4) and comparing the exponents, we find that $\alpha=-2$ and the function u_0 is given by

$$u_0\ -6\phi_x^2. \tag{7}$$

Next we determine the resonances. The values of j are called resonances where arbitrary functions of x and t can be introduced into the expansion. Inserting the ansatz (5) together with $\alpha=-2$ into (4) we find

$$\begin{aligned}\phi_x\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}&(j-2)u_ju_k\phi^{j+k-5}+\tfrac{1}{2}\phi_x^3\sum_{j=0}^{\infty}(j-2)(j-3)(j-4)u_j\phi^{j-5}\\
&+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}u_{jx}u_k\phi^{j+k-4}+\tfrac{3}{2}\phi_x^2\sum_{j=0}^{\infty}(j-2)(j-3)u_{jx}\phi^{j-4}\\
&+\tfrac{3}{2}\phi_x\phi_{xx}\sum_{j=0}^{\infty}(j-2)(j-3)u_j\phi^{j-4}+\phi_t\sum_{j=0}^{\infty}(j-2)u_j\phi^{j-3}\\
&+\tfrac{1}{2}\phi_{xxx}\sum_{j=0}^{\infty}(j-2)u_j\phi^{j-3}+\tfrac{3}{2}\phi_{xx}\sum_{j=0}^{\infty}(j-2)u_{jx}\phi^{j-3}\\
&+\tfrac{3}{2}\phi_x\sum_{j=0}^{\infty}(j-2)u_{jxx}\phi^{j-3}+\tfrac{1}{2}\sum_{j=0}^{\infty}u_{jxxx}\phi^{j-2}\\
&+\sum_{j=0}^{\infty}u_{jt}\phi^{j-2}+(a/t)\sum_{j=0}^{\infty}u_j\phi^{j-2}=0.\end{aligned} \tag{8}$$

The resonances m are determined from the coefficients with the factors ϕ^{j+k-5} and ϕ^{j-5}. For the coefficient with the factor ϕ^{j+k-5} we have to put $j=0$, $k=m$ and $j=m$, $k=0$. For the coefficient with the factor ϕ^{j-5} we have to put $j=m$. Then we find that

$$0=(m-2)\phi_xu_mu_0-2\phi_xu_0u_m+\tfrac{1}{2}(m-2)(m-3)(m-4)\phi_x^3u_m. \tag{9}$$

Inserting (7) into (9) we find that

$$0=-6(m-4)+\tfrac{1}{2}(m-2)(m-3)(m-4). \tag{10}$$

Thus the resonances are given by $m_1=-1$, $m_2=4$ and $m_3=6$. The value $m_1=-1$ corresponds to the arbitrary (undefined) singularity manifold ($\phi=0$). We find the same result as for the KdV equation. The additive term au/t does not change the resonances.

Solving (8) we find that

$$j=0 \qquad u_0=-6\phi_x^2 \tag{11}$$

$$j=1 \qquad u_1=6\phi_{xx} \tag{12}$$

$$j=2 \qquad \phi_x\phi_t+\phi_x^2u_2+2\phi_x\phi_{xxx}-\tfrac{3}{2}\phi_{xx}^2=0 \tag{13}$$

$$j=3 \qquad \phi_{xt}+\phi_{xx}u_2-\phi_x^2u_3+\tfrac{1}{2}\phi_{xxxx}+(a/t)\phi_x=0 \tag{14}$$

$$j=4 \qquad \text{compatibility condition}$$

$$(\phi_{xt}+\tfrac{1}{2}\phi_{xxxx}+\phi_{xx}u_2-\phi_x^2u_3+(a/t)\phi_x)_x=0. \tag{15}$$

Thus, if (14) is satisfied, then (15) is satisfied and in this case the 'coefficient' u_4 is arbitrary. For $j=5$ we obtain

$$\begin{aligned}-3\phi_x^3u_5&=(\phi_xu_1+3\phi_x\phi_{xx})u_4+(u_0+3\phi_x^2)u_{4x}+u_{0x}u_4+(\phi_t+\tfrac{1}{2}\phi_{xxx}+\phi_xu_2+u_{1x})u_3\\&\quad+(u_1+\tfrac{3}{2}\phi_{xx})u_{3x}+\tfrac{3}{2}\phi_xu_{3xx}+u_{2t}+u_2u_{2x}+\tfrac{1}{2}u_{2xxx}+(a/t)u_2.\end{aligned} \tag{16}$$

For $m\geqslant 3$ the recursion relation for the functions u_j is given by

$$\begin{aligned}&(m-4)(\phi_t+\tfrac{1}{2}\phi_{xxx})u_{m-2}+u_{(m-3)t}+\phi_x\sum_{j=1}^{m-1}(j-2)u_ju_{m-j}\\&\quad+\sum_{j=0}^{m-1}u_{jx}u_{m-j-1}+\tfrac{3}{2}(m-4)(\phi_{xx}u_{(m-2)x}+\phi_xu_{(m-2)xx})\\&\quad+\tfrac{3}{2}(m-4)(m-3)(\phi_x^2u_{(m-1)x}+\phi_x\phi_{xx}u_{m-1})+\tfrac{1}{2}u_{(m-3)xxx}+(a/t)u_{m-3}\\&=\tfrac{1}{2}(m+1)(m-4)(m-6)\phi_x^3u_m.\end{aligned} \tag{17}$$

It follows that if $u_3=u_4=u_6=0$ and the function u_2 satisfies the cKdV equation, then $u_5=0$ and all other functions u_j ($j\geqslant 7$) vanish. Thus we have obtained the following overdetermined system of partial differential equations:

$$\phi_x\phi_t+\phi_x^2u_2+2\phi_x\phi_{xxx}-\tfrac{3}{2}\phi_{xx}^2=0 \tag{18a}$$

$$\phi_{xt}+\phi_{xx}u_2+\tfrac{1}{2}\phi_{xxxx}+(a/t)\phi_x=0 \tag{18b}$$

$$u_{2t}+u_2u_{2x}+\tfrac{1}{2}u_{2xxx}+au_2/t=0 \tag{18c}$$

$$u=-6\phi_x^2\phi^{-2}+6\phi_{xx}\phi^{-1}+u_2. \tag{18d}$$

If (18a)–(18c) are compatible (there are three equations for two fields), then we have found a Bäcklund transformation. To prove this, we set

$$\phi_x=V^2 \tag{19}$$

and find by straightforward calculation that

$$V_t+2V_{xxx}+u_2V_x+\tfrac{1}{2}u_{2x}V=0 \tag{20a}$$

$$V_t+\tfrac{1}{2}V_{xxx}+\tfrac{3}{2}V_xV_{xx}V^{-1}+u_2V_x+(a/2t)V=0. \tag{20b}$$

Eliminating V_t we find from (20*a*) and (20*b*)

$$\tfrac{3}{2}V_{xxx}-\tfrac{3}{2}V_xV_{xx}V^{-1}=(a/2t)V-\tfrac{1}{2}u_{2x}V. \tag{21}$$

This equation can be written as

$$\tfrac{3}{2}(V_{xx}V^{-1})_x=(a/2t)-u_{2x}/2. \tag{22}$$

Integration yields

$$V_{xx}V^{-1}=ax/(3t)-u_2/3+\lambda(t). \tag{23}$$

Equation (23) can be written as

$$(f(t)D^2+f(t)u_2/3-axf(t)/(3t))V=\mu V, \tag{24}$$

where we have put $\lambda(t)=\mu/f(t)$. We thus have found a candidate for a Lax formulation of (18*c*) by defining the operators

$$L(u_2,t)=f(t)(D^2+u_2/3-ax/3t) \tag{25a}$$

$$B(u_2)=2D^3+u_2D+\tfrac{1}{2}u_{2x}. \tag{25b}$$

Equations (24) and (20*a*) then read

$$LV=\mu V \tag{26a}$$

$$V_t=-BV. \tag{26b}$$

The eigenvalue problem (26*a*) is compatible with the time evolution of the eigenfunction V given by (26*b*), if we have the operator identity

$$\mathrm{d}L/\mathrm{d}t=LB-BL, \tag{27}$$

where $\mathrm{d}/\mathrm{d}t$ denotes the derivative with respect to both the explicit time dependence of L and the implicit dependence via $u_2(x,t)$. Checking (27) with (25*a*) and (25*b*) we find compatibility only for the two cases $a=0$, $f(t)=1$ or $a=\tfrac{1}{2}$, $f(t)=t$. The case $a=0$ corresponds to the KdV equation and $a=\tfrac{1}{2}$ to the cKdV equation. Consequently, (18*a*)–(18*d*) define a Bäcklund transformation for the value $a=0$ and $a=\tfrac{1}{2}$ and (25*a*) and (25*b*) are the Lax representations.

To summarise: we have shown that the Lax representation and a Bäcklund transformation can be obtained for the cKdV equation in a remarkably simple manner from the Painlevé property.

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